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A STUDY OF RECENT NUMERICAL METHODS
FOR THE BAROTROPIC PRIMITIVE EQUATIONS

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In recent months there has been considerable interest in applying finite element methods to time-dependent problems in meteorology and oceanography. This paper analyzes a number of recent papers dealing with wave propagation in non-linear equations with the purpose of delineating some of the more obvious mathematical problems which must be addressed regarding the use of finite elements in numerical forecasting. Some new results are presented.

1. Introduction

In doing numerical weather forecasting there are essentially two problems: 1) the forecasting, which amounts to solving a coupled set of non-linear partial differential equations in space and time, and 2) determining the initial conditions for those equations from observational data. The problem of determining the initial conditions accurately is a very serious one which has not yet been approached by finite element methods. Thus we shall not discuss it in this report.

The forecast equations are essentially the laws of conservations of mass, momentum and energy. Written in the usual form, as a set of first order partial differential equations, they are called the "Primitive Equations" or the "P.E. model." As such they are hyperbolic, and non-linear. These equations are

$$\left(\frac{\partial}{\partial t} + V\cdot\nabla + w\,\frac{\partial}{\partial z}\right)\vec{V} + F\,\hat{k} \times \vec{V} + \frac{1}{\rho}\,\nabla p = \nu\nabla^2\vec{V} \qquad (1a)$$

$$g + \frac{1}{\rho}\,\frac{\partial p}{\partial z} = 0 \qquad (1b)$$

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla + w\,\frac{\partial}{\partial z}\right)\log\,\Theta = \frac{\dot{Q}}{C_P T} \qquad (1c)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) + \frac{\partial}{\partial z}\,(\rho w) = 0 \qquad (1d)$$

where F is the Coriolis "parameter" $2\,\Omega\,\sin\,\theta$, Θ is the potential temperature $(\Theta = T(P/P_o)^{-R/C_P})$, $\vec{V}$ is the horizontal velocity vector and w is the vertical velocity. The wave solutions which result consist of two types of waves - the so-called inertial-gravity waves and the Rossby waves (See [1]). It is known that the gravity waves should not be present, and in any numerical computation they die out, with time constants of 12-24 hours. However, this length of time is the time of primary interest for forecast purposes.

The laws of conservation can also be written in terms of scalar and vector potentials i.e., a geopotential ϕ, a vorticity ξ and a stream function ψ. These functions satisfy a second order elliptic and a single hyperbolic equation and thus have the property that the gravity waves are damped out, or filtered. (See equations (27) and (29)). This form of the equations is referred to as the vorticity, or filtered, form of the equations. However, involving a second order equation they require different boundary conditions, which conditions are somewhat contrived.

For the primitive equations (1), one can set up an order of magnitude analysis and determine which terms are significant for meteorological purposes. This leads to a hierarchy of equations which one can solve more easily than (1). The zero order analysis gives the following set

$$F\,\hat{k} \times \vec{V} = -\frac{1}{\rho}\,\nabla p \tag{2a}$$

$$\nabla\cdot(\rho\vec{V}) = 0, \tag{2b}$$

which dominate all the other terms by an order of magnitude. This suggests that the basic flow is horizontal, divergence free, and driven by the balance of the pressure force and the Coriolis force.

At the next level we have the so called barotopic equations

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)\vec{V} + F\,\hat{k} \times \vec{V} + \nabla\phi = 0 \tag{3a}$$

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)\phi + \phi\nabla\cdot\vec{V} = 0 \tag{3b}$$

where $\phi = gh$ is the geopotential at the height h, obtained by integrating $\frac{1}{\rho}\,dp$ in the z direction, since equations (3) are independent of vertical variables.

If we take a perturbation approach about some uniform velocity $\vec{V}_o$ and a height H, we get a set of hyperbolic first order equations

$$\frac{\partial u}{\partial t} = - \vec{V}_o \cdot \nabla u + F\, v - \frac{\partial \phi}{\partial x} \tag{4a}$$

$$\frac{\partial v}{\partial t} = - \vec{V}_o \cdot \nabla v - Fu - \frac{\partial \phi}{\partial y} \tag{4b}$$

$$\frac{\partial \phi}{\partial t} = - \vec{V}_o \cdot \nabla \phi - gH_o\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4c}$$

where $\vec{V} = \hat{i}u + \hat{j}v$. If we look for a solution of the form

$$\psi = (u,v,\phi) = \psi_o e^{i(\omega t - k \cdot x)} \tag{5}$$

we find three waves

$$\omega_o = \vec{V}_o \cdot \hat{k} \tag{6a}$$

$$\omega_{1,2} = \vec{V}_o \cdot \hat{k} \pm \sqrt{gH_o \hat{k} \cdot \hat{k} + F^2} \tag{6b}$$

The first is a convective wave, due to the uniform flow. The second and third are "inertial-gravity" waves. If the terms $(Fv, -\frac{\partial \phi}{\partial x})$ were zero in (4a) and $(-Fu, \frac{\partial \phi}{\partial y})$ were zero in (4b), then these twc waves would disappear. These are just the terms in (2a), which essentially "balance out."

If the initial conditions contain any of the eigenvectors to which these last two waves are the eigenvalues, then the linear solution contains them. Thus any solution to the non-linear problem which initially contains some component in these "directions" will propagate them for some time. The real problem is non-linear, and in actuality they die out but we would hope that they are not there to start with.

As a result there is considerable interest in getting correct initial conditions from the observational data. We turn first to the methods which involve vorticity. These all involve the assumption that there is a static balance of the wind and pressure fields, and that one can be derived from the other. Proceeding from there is complicated by the fact that in the tropics there are very small pressure changes, so that observational <u>errors</u> in pressure can be as great as the pressure changes themselves, while the winds tend to be more accurately recorded. The converse is true in the mid - and high - latitudes. So in the mid-latitudes one takes the pressure measurements at various points (invariably <u>not</u> grid points), computes the pressure at the grid points by some weighted average, converts to the geopotential ϕ, and then computes the stream function ψ using one of the three relations (9) - (11). The winds are then obtained from ψ. In the tropics the proceedure is reversed. The observed winds $\vec{V}$ are used to compute the vorticity

$$\xi = \hat{k} \cdot \nabla \times \vec{V} \tag{7}$$

The stream function ψ is obtained from the Poisson equation,

$$\nabla^2 \psi = \xi \tag{8}$$

Then the geopotential ϕ is obtained by solving one of the following:

$$\nabla^2 \phi = F \nabla^2 \psi \tag{9}$$

$$\nabla^2\phi = \nabla\cdot(F\nabla\psi) \tag{10}$$

$$\nabla^2\phi = 2J\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) + \vec{\nabla}\cdot(F\nabla\psi) \tag{11}$$

where F is again the Coriolis factor,

$$F = 2\Omega \sin\theta \tag{12}$$

J is the symbol for the Jacobian, and θ is the latitude from the equator.

These equations are called the quasi-geostrophic relation, the linear balance equation, and the non-linear balance equation. Note from (12) that to use (9), (10) or (11) to solve for ψ at the equator involves solving a singular differential operator. Note also that these equations are all Poisson equations. Note also that this procedure only deals with one part of the wind, the rotational part. Nowhere does the divergent part enter the calculation. Since the wind is assumed non-divergent in the first place, that is not essential at the start. However, a finite difference scheme will introduce some errors, and this may be significant.

2. Finite Difference Models

These methods were used, and are still used in some places, for many years. However, more and more people are turning to the primitive equations. So let us consider the simplest version, the barotropic model. First we note that the gravity waves (6b) have a much higher speed (about 300 meters/second) than the meteorological mode (6a) (5 meters/second). For <u>computing</u>, using a system of hyperbolic equations, the Courant-Lewy-Fredericks criterion requires that the time step be less than Δx divided by the wave speed. These fast waves then dominate the size of the time step, for calculating using an explicit calculation. Thus there is considerable interest in semi-implicit schemes.

Williamson and Browning [2] run some computer tests on the primitive equations, using the barotropic model, writing the equations in spherical coordinates. They use both the advection form, (a is the Earth's radius)

$$\frac{\partial u}{\partial t} = -\frac{u}{a\cos\theta}\frac{\partial u}{\partial\lambda} - \frac{v}{a}\frac{\partial u}{\partial\theta} + (f + \frac{u}{a}\tan\theta)v - \frac{1}{a\cos\theta}\frac{\partial\phi}{\partial\lambda} \tag{13}$$

$$\frac{\partial v}{\partial t} = -\frac{u}{a\cos\theta}\frac{\partial v}{\partial\lambda} - \frac{v}{a}\frac{\partial v}{\partial\theta} - (f + \frac{u}{a}\tan\theta)u - \frac{1}{a}\frac{\partial\phi}{\partial\theta} \tag{14}$$

$$\frac{\partial\phi}{\partial t} = -\frac{1}{a\cos\theta}\{\frac{\partial}{\partial\lambda}(\phi u) + \frac{\partial}{\partial\theta}(\phi v\cos\theta)\} \tag{15}$$

and the flux form of (13) and (14)

$$\frac{\partial(\phi u)}{\partial t} = -\frac{1}{a\cos\theta}\{\frac{\partial}{\partial\lambda}(\phi u^2) + \frac{\partial}{\partial\theta}(\phi\, uv\cos\theta)\} + (f + \frac{u}{a}\tan\theta)\phi v - \frac{1}{a\cos\theta}\frac{\partial}{\partial\lambda}(\phi^2/2) \tag{16}$$

$$\frac{\partial}{\partial t}(\phi v) = -\frac{1}{a\cos\theta}\{\frac{\partial}{\partial\lambda}(\phi\, uv) + \frac{\partial}{\partial\theta}(\phi v^2\cos\theta)\} - (f + \frac{u}{a}\tan\theta)\phi u - \frac{1}{a}\frac{\partial}{\partial\theta}(\phi^2/2) \tag{17}$$

In their numerical scheme they use a centered time difference and a centered space difference for all derivatives. For the terms involving $(f + \frac{u}{a}\tan\theta)$ they use a time average $(u = \frac{1}{2}[u(t+1, x) + u(t-1, x)])$ so that these terms in either (13) and (14) or (16) and (17) are treated implicitly - thus creating a semi-implicit model.

They ran three tests. In the first, they used 5° grids in both λ and θ, taking the time step based on the smallest spacial distance near the pole. In the second they arbitrarily omitted points on longitude circles as the latitude increased towards the pole, so that at 60°N they had 72 points per longitude circle and at 85°N they had 12. (This has the same effect as an icosohedral grid). This allows a longer time step, by a factor of 6. In the third, they took the output, did a longitudinal Fourier analysis and threw away the high frequency, short wave components ("filtering"). (This is what is proposed for FNWC P.E. model). They start up the scheme with a known steady state solution of (13) - (15), namely zonal geostrophic flow, given by

$$
\begin{aligned}
u &= u_o(\cos\theta \cos\alpha - \cos\lambda \sin\theta \sin\alpha) \\
v &= u_o \sin\lambda \sin\alpha \\
\phi &= \phi_o - (a\,\Omega\, u_o + \frac{u_o^2}{2})(\cos\lambda \cos\theta \sin\alpha + \sin\theta \cos\alpha)^2
\end{aligned}
\tag{18}
$$

(If $\alpha = \pi/2$ there is no tendency to flow across the equator, which simplifies the calculation).

Their results are - for test one

1) For the initial time step the error in u is dominated by the truncation error in ϕ.

2) The error in u grew by two orders of magnitude in five days, from the error after one time step.

3) The errors start at the pole and spread out.

In test two, there was

1) a factor of ten in the size of the error near the poles.

In test three they used the same time step as in test two (six minutes vs one minute for test one). The computations were three times faster than in test one. They were able to reproduce exactly the results of test one. With a <u>fourth</u> order difference scheme in space and the filtered scheme they were able to get the best results by about two orders of magnitude. Thus for this type calculation a filtered model is clearly the best.

Kwizak and Roberts [3] rewrite the three equations (13) - (15) as follows:

Let

$$K \equiv \frac{1}{2}\left(u^2 + v^2\right) \tag{19}$$

and

$$Q \equiv F + \frac{1}{a\cos\theta}\left[\frac{\partial}{\partial\lambda} v - \frac{\partial}{\partial\theta}(u \cos\theta)\right]. \tag{20}$$

Then we can write

$$\frac{\partial u}{\partial t} = - \frac{1}{a\cos\theta}\frac{\partial\phi}{\partial\lambda} + Qv - \frac{1}{a\cos\theta}\frac{\partial K}{\partial\lambda} \tag{21}$$

$$\frac{\partial v}{\partial t} = - \frac{1}{a}\frac{\partial\phi}{\partial\theta} - Qu - \frac{1}{a}\frac{\partial K}{\partial\theta} \tag{22}$$

$$\frac{\partial\phi}{\partial t} = - \frac{1}{a\cos\theta}\left\{\frac{\partial}{\partial\lambda}(\phi u) + \frac{\partial}{\partial\theta}(\phi v \cos\theta)\right\} \tag{23}$$

The function K is interpretable as the kinetic energy and Q is interpretable as the absolute vorticity [3] or the potential vorticity [4]. The latter comes from the fact that (20) can be written

$$Q = F + \nabla \times \vec{V} \tag{24}$$

where $\vec{V}$ is the velocity vector.

Now the original equations are derivable from assuming an incompressible gas, so that we have the implicit equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{25}$$

which implies that there is a stream function ψ such that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \tag{26}$$

so that (23) can be rewritten

$$\nabla^2 \psi = Q - F \tag{27}$$

And taking the curl of (21) - (22) we get

$$\frac{\partial Q}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial Q}{\partial \lambda} + \frac{v}{a}\frac{\partial Q}{\partial \theta} = 0 \tag{28}$$

which can be rewritten

$$\frac{\partial Q}{\partial t} = J(Q,\psi) \tag{29}$$

where J is the Jacobian of Q and ψ. The pair (26) and (29) are a coupled system, with (26) being an elliptic equation, which could be solved iteratively, and then ϕ could be found from (15).

Observe that taking the curl of (21) - (22) eliminates the ϕ from these equations. Typical values of the variables in the atmosphere are $\phi \approx 3 \times 10^4$, $u = 5$, $v = 1$. So a potentially large term has been eliminated from the computations for the velocity $\vec{V}$. Also observe that (28) is the only wave-like equation left for $\vec{V}$ and this, being first order, has only one wave solution. A linearization of (28) about a steady flow V_o will give only one wave

$$Q \sim e^{ik(x-V_o t)}$$

which is the "meteorological" wave which is expected: that is, the two gravity waves do not enter into the computation for Q. This is the reason that the vorticity form of the equations was used for many years - the time dependent portion of the problem involved only a single wave speed, the one which was desired. The problem is that the solution must be kept divergence free; this may be a nontrivial task. For experimental purposes we can see that the "meteorological wave" ought to be associated with having a divergence free field, and thus making an effort to get the initial conditions divergence-free is a worthwhile computation, as far as reducing "unwanted noise" in the answer. For this reason Kwizak and Roberts comment "... the winds are perfectly non-divergent initially and at the end of the first time step. This property virtually eliminates the gravity waves from the integration," [3].

The computations which Kwizak and Roberts do are based on the three primitive equations (21), (22) and (15), doing a semi-implicit scheme. They take the ϕ terms in (21) and (22), and then rewrite (15) as

$$\phi_t = \frac{\phi_o}{a\cos\theta}\{\frac{\partial}{\partial\lambda} u + \frac{\partial}{\partial\theta}(v \cos\theta)\} - \frac{1}{a\cos\theta}\{\frac{\partial}{\partial\lambda}([\phi-\phi_o]u) + \frac{\partial}{\partial\theta}([\phi-\phi_o]v \cos\theta)\} \quad (30)$$

and then treat the first term implicitly. They thus have three coupled equations to handle. The actual method of computation is to observe that (21) and (22) can be used to convert (29) to a non-homogeneous Helmholtz equation for $\bar{\phi}^{2t}$, defined as $\bar{\phi}^{2t} \equiv [\phi(t+\Delta t) + \phi(t-\Delta t)]$, namely

$$\frac{1}{a\cos\theta}(\Delta t)^2\phi_o\nabla^2\bar{\phi}^{2t} - \bar{\phi}^{2t} = f(\phi,u,v) \quad (31)$$

Having found this, then (21) and (22) can be used to compute u and v in a purely explicit fashion. For their computations they can use a 60 minute time step (vs a 10 minute for an explicit calculation) at a savings of 4 times in the computation speed. Thus the method of computation does <u>not</u> involve three coupled implicit equations but one Helmholtz equation (second order) and two explicit calculations.

Two years later Elviers and Sundstrom [5] do a similar test, using essentially the same scheme, noting that the averaging operator which is used allows for a decoupling of the equations for even and odd time steps, i.e., a staggered grid. They do a stability analysis of the finite difference models. Their analysis shows the semi-implicit scheme is superior to an explicit scheme.

Williamson and Browning [2] and Kwizak and Roberts [3] both do a semi-implicit scheme, but they differ as to which terms they treat implicitly. Kwizak and Roberts do the more usual method. In the u and v equations they treat the pressure gradient term (involving ϕ) implicitly. This allows them to eventually eliminate u and v from the ϕ equation and convert the latter to a second order Helmholtz equation. This one equation is solved fully implicitly, with u and v then found explicitly. Williamson and Browning treat the Coriolis terms implicitly, giving a coupled system for u and v, while the ϕ equation is treated explicitly. In both cases only one of the terms driving the gravity waves is treated implicitly, but this is sufficient to remove them from the Courant criterion.

Thus a significant problem for the predictive equations is the ability to solve quickly and efficiently a system of partial differential equations. The usual method is to use classical finite difference techniques. The standard way is to use second order centered differences. There is every reason to believe that fourth order centered differences in space will increase the accuracy. This is currently being investigated [6].

3. Finite Element Methods

A question of definite interest is whether the "finite element method" of solving partial differential equations will give better accuracy. I am aware of only four places where this is currently being investigated. George Fix [4,7] is studying ocean circulation problems this way. His studies are being continued by Hirsch [17]. M. P. Cullen [8, 9, 13] has programmed the barotropic equations, and is now attempting to program a more realistic set of primitive equations. A. Staniforth [10] is attempting to implement finite element calculations in the Canadian Meteorological Office. And a student thesis at NPS by Donald Hinsman [11] ran some experiments with finite elements on the barotropic equations. All of these indicate that this method may have a significant future in meteorology.

Fix [4] looks at the ocean circulation problem, which is just (13) and (14), together with the divergence free condition

$$\nabla \cdot \vec{V} = 0$$

Thus he does not get involved with gravity waves and has only the "meteorological mode" to contend with. He then converts to the vorticity equation and the non-linear balance equation (27), (29). He then takes a finite element - Galerkin approach, using linear elements (also quadratics and cubics for further tests).

There are three problems to be addressed in any analysis of this discretization. The first is the accuracy of the spacial discretization. The second, which is due to the fact that these equations are non-linear, is called "aliasing," a feature which can be most easily seen by considering a Fourier analysis of the spacial terms. If both u and v are written as Fourier series and then truncated at the $N^{\underline{th}}$ harmonic, then a term involving u times v will have a term, say $(u_N \cos N x)(v_M \cos M x)$, which would normally give rise to two terms $\frac{1}{2} u_N v_M \cos(M-N)x$ and $\frac{1}{2} u_N v_M \cos(N+M)x$, but the latter can not appear due to the truncation. Certain discretization schemes have an imbalance in the treatment of this phenomenon, known as aliasing. However, Jesperson [12] has shown that this phenomenon does <u>not</u> occur with a finite element scheme, that is, finite element schemes are free from aliasing, a fact which Fix reconfirms.

Fix also shows what is widely known, that the spacial accuracy for the velocity is $O(h^k)$ where h is the grid size and k is 1, 2, or three depending upon whether linear, quadratic or cubic elements are used.

The third problem is how to handle the time integration. Fix does not have to worry about a semi-implicit scheme from the point of view of gravity waves. However, any finite element scheme links more than two points and one is automatically <u>forced</u> into an implicit scheme. That is one drawback for finite elements. Fix proceeds to analyze a <u>linear</u> one dimensional analogue of the wave propogation problem. The finite element method generates its own natural set of difference equations. For the linear problem which Fix sets up, using linear elements in space with time varying coefficients, Fix shows that the implicit discretization which the finite element method forces gives fourth order accuracy in the phase speed of waves, a fact which Cullen had noted earlier [8].

Fix then chooses for his time discretization the usual centered difference (leap-frog) time discretization, so that the final computation for Q is given by

$$\iint_A [Q^{n+1} - Q^{n-1}]\phi_j \, dA = 2\Delta t \iint_A J(Q^n, \psi^n) \, dA \ .$$

Thus for the non-linear ocean circulation problem one knows that the finite element method does not introduce aliasing, is spatially as accurate as we make the finite elements, and can be conjectured to be fourth order accurate in phase speed. (Phase speed of the waves has always been a problem in forecasting of weather).

In a series of three papers [8], [9], [13], Cullen tackles two problems. The first problem is to solve equations (3) in a limited area $-1 \leq x, y \leq 1$ (the "beta plane") with periodic boundary conditions, using a finite element method. The second is to solve equations (3) on the surface of the globe. His analysis proceeds as follows.

He first [13] considers a single linear equation

$$\frac{\partial\phi}{\partial t} + \vec{V}\cdot\nabla\phi = 0 \qquad (32)$$

on a rectangle, using rectangles and bilinear elements, with V known. He compares his results, using a 16 x 16 grid, to second and fourth order finite difference schemes using a 32 x 32 grid. The exact solution to this problem is known. He demonstrates that the finite element calculation is more accurate, both in handling smooth data and in handling a problem with a discontinuity.

Satisfied with the results of the linear problem, Cullen [13] proceeds to the non-linear problem (3), in a grid $-1 \leq x, y \leq 1$, <u>without</u> the Coriolis term. His initial condition would give a gravity wave if the problem were linear. He compares his answer with a 16 x 16 grid to finite difference schemes with 32 x 32 and 64 x 64 grids. He uses linear finite elements with a leap-frog time step. His results indicated that the finite element method was better, although there is no exact solution to compare with. Cullen then attempts to analyze the numerical scheme. For the linear one dimensional case he shows that the phase error is fourth order in $\Delta t/\Delta x$. In fact, he shows something more, namely that for some range of $\Delta t/\Delta x$ the phase error can actually cause a small leading phase. He also argues, largely on qualitative grounds, that there is no aliasing problem. There is no discussion of the inertial vs Rossby waves, and no attempt to isolate one from the other, or to control either.

Cullen's second paper [8] concerns equations (3) on the entire globe - the genuine barotropic problem. One question which Cullen now addresses is how is the best way to handle the non-linear terms which appear in (3). His approach is to analyze a single one dimensional term of the form $u \frac{\partial v}{\partial x}$. The various possibilities are:

1. Treat v as known, take the nodel values of v, compute a derivative, and simply multiply the coefficients for u by these values.

2. Compute a finite element expansion for v, analytically differentiate this expansion to find $\frac{\partial v}{\partial x}$, and use the resultant two expansions to compute $t = u \frac{\partial v}{\partial x}$.

3. Compute a finite element expansion for $\frac{\partial v}{\partial x}$ itself.

Cullen claims that the last method, while sacrificing some accuracy, controls aliasing much better than either of the first two.

I was unable to follow his arguments, and will attempt in a further report to see whether it can be generalized. In any event Cullen uses the third method in his calculations [8].

In [8] Cullen runs four tests on the equations (3), using the three methods above (for two of the runs he slightly modifies the coefficients in method 3). He takes as initial values a finite element solution to (2a), comparing with other published results using finite differences. Assuming that the published finite difference results are the most accurate (?) he notes that the phases on most finite element runs appear to lag, although some are advanced. He concludes that:

(a) Waves down to four element lengths will be treated almost perfectly.

(b) Waves less than two grid lengths will not be treated at all well.

(c) The finite element scheme "essentially eliminates aliasing errors."

(d) The boundary condtions used on the problem introduced errors of the same order of magnitude as the change from finite differences to finite elements, and are thus quite significant.

In his third paper [9] Cullen reports on actual computations on a sphere, relying heavily on the analysis above. He concludes "the finite element method is computationally more efficient." He uses an icosahedral grid, subdivided by latitude and longitude lines to form triangles, resulting in 1002 points. In integration he treats the trig functions in

(3) as constants over each small triangle. He uses the scheme of [8] to treat the non-linear terms.

For his actual computational scheme it appears that the only implicit portion of the scheme is the implicitness generated by the left hand side. In other words, looking at (3) as equations of the form

$$\frac{\partial \psi}{\partial t} = F(\psi)$$

he solves the resultant non-linear equations by the following iterative process. Treat the right hand side as known and take the diagonally dominant matrix on the left as the generator of the next iteration. In this way the iterative process to find the values of the nodal points at time t is relatively fast. Then a leap-frog time discretization of 10 minutes was used. (He could use one of up to 14 minutes). He found that filtering was required every two hours to get long time (greater than 5 days) solutions. His initial conditions were Rossby waves with wave number 4 and wave number 8. He compares his results with published results using a finite difference model with 4032 points, one with 14,592 points and a spectral model with 640 degrees of freedom. His results are better than the first but not as good as the last two. He also observes that the errors seem to start from the vertices of the icosahedron, where there are only 5 supporting triangles instead of 6 as there are at the intermediate triangulation points.

Hinsman [11] in his master's thesis again studied equations (3). He considered two possible grids. One used the two angles λ and θ as rectangular coordinates and triangulating the resultant "rectangle." This results in a very fine subdivision of the polar regions resulting in very small physical lengths Δx in that region. The second grid was an icosahedral grid, (as did Cullen [8], [9]) which was subsequently subdivided by arcs of great circles. This results in most of the nodes having 6 adjacent nodes. The corners of the icosahedron have only 5 adjacent nodes. This appears to generate some "noise," as Cullen noted [9].

Experiments with the (λ,θ) grid showed instability after 12 hours, as might be guessed from the fact that there were 36 points around each latitude circle including those of the poles. The instability clearly arose in the polar region. Experiments with the icosahedral grid did not show these problems.

Starting with an analytic solution to the non-linear balance equation which has essentially one wave going around the earth, Hinsman Fourier analyzed the solution as it propagated, comparing his results to similar finite difference calculations. In the low latitudes the propagation speed was almost exactly correct; as compared to 50-60% correct for finite differences. In the high latitudes both finite elements and finite differences fall behind the predicted speed. (This contradicts Cullen's observations).

The method of solution was very different from any previous methods. Each equation in (3) is "quasi-linearized" by considering the other two variables as known (from a previous calculation). The finite element scheme automatically generates an implicit scheme, but the decision was made to go one step further and use a Crank-Nicholson approach, essentially calculating the variables at time step (N+1/2). The quasi-linearization uncoupled the equations, but, like Cullen, there was no attempt to distinguish between the Rossby wave and the gravity waves. Finite elements were used to expand all the variables, having first integrated by parts to get the weak form. The equations were then successively solved, first for ϕ, then for u and finally for v. (The trig functions $\sin\theta$ and $\cos\theta$ were also written as finite elements, as opposed to Cullen who treated them as constants over triangles). The resultant equations were solved by a Gauss-Seidel iteration. This took about 10-12 iterations to converge. (An additional feature of the program was a very efficient coding scheme to avoid completely the storage of the zero elements of the matrices involved).

Swartz and Wendroff [14] compare the relative efficiency of finite difference and finite element methods. They do so by taking a one dimensional linear problem. Their interest is somewhat different than ours and thus their data is recorded very differently. They record, for example, (Table 1) the number of intervals per wave length which are necessary to get a desired accuracy in phase (in theory). One conclusion which can be drawn is that a dramatic improvement in resolution can be obtained by switching from linear elements to quadratic elements (for an error of 10^{-2} the number of intervals per wavelength goes from 8.7 to 4.8, while for 10^{-4} error the change is from 27 to 9.7).

They also, using results of Kreiss and Oliger, report the results of finite difference theory. For fourth order spacial accuracy, - which is what <u>linear</u> elements give - the corresponding results are a change from 13.3 to 7.9 and from 42.5 to 17.3. So one can conclude that in theory the finite elements have better phase resolution (e.g., 8.7 intervals per wave length vs 13.3 for linear elements and 4th order differencing) and that the payoff for increasee complexity is also greater.

The method that they use for actually computing the finite element solutions is interesting - they call it "use of the trapezoidal rule." Consider the equation

$$\frac{\partial u}{\partial t} = c \frac{\partial u}{\partial x}$$

and compute the finite spacial element expansion of this to get a formula

$$A \dot{\tilde{u}} = B \tilde{u} \tag{33}$$

where A and B are sparse matrices and $\tilde{u}$ is a vector (for linear elements A has a $\frac{1}{6}$ (1, 4, 1) structure). Now take a time discretization which gives

$$A \frac{\tilde{u}^{n+1} - \tilde{u}^{n}}{\Delta t} = B \frac{\tilde{u}^{n+1} + \tilde{u}^{n}}{2} \tag{34}$$

This scheme is fully implicit and is stable for all h, Δt. They compare this with the leap-frog finite difference scheme.

Before finishing with the linear problem they conclude that a <u>twelfth</u> order spacial difference scheme, with leapfrog, is competitive with a cubic spline, for phase errors of 10^{-4}.

They turn to the first order one-dimensional non-linear problem

$$u_t = \frac{\partial}{\partial x} f(u) + g(u) \ . \tag{35}$$

They have two conclusions, one of which is given just in passing. They remark that they will not use finite elements directly on (41). Instead they will use the operator $A^{-1}B$ to replace $\frac{\partial}{\partial x}$, where A and B are defined in (33). Their reasoning is that the finite element scheme is awkward and that it has been shown that use of the finite element scheme directly will degrade the spacial accuracy [15]. The scheme above will keep the truncation error. They further remark that "It no longer makes sense to try to find the best scheme. A reasonable approach seems to be to compare schemes using the same number of intervals per wave length." This sort of reasoning is directly counter to what Cullen and Hinsman have used.

Their second conclusion, based on the above and counting evaluations of $f(u)$ and $g(u)$ as the major source of computer time, is that, using cubic splines and <u>18th</u> order differencing as <u>comparable</u> schemes, finite difference and finite element schemes are roughly comparable. (For global meteorology an 18th order difference scheme seems excessive).

Another paper which deals with non-linear finite element calculations for hyperbolic problems is by Oden and Fost. What Oden and Fost do to find the finite element formula for a hyperbolic problem is to use as a test function $\dot{V}_i \left(\frac{\partial V_i}{\partial t}\right)$, so that the equation with which they work is the analogue of $\frac{dE}{dt} = 0$, where E is the energy of the system. It is then clear that their method preserves energy. In the actual computation they use the centered (leapfrog) approximation to $\frac{\partial V_i}{\partial t}$, and their analysis is of a single second order equation

$$\frac{\partial^2 u}{\partial t^2} - c \frac{\partial T}{\partial u_x}(u_x) \frac{\partial^2 u}{\partial x^2} = 0 , \tag{36}$$

which, in weak form, is

$$\left(\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right) + c^2 \left(T(u_x), \frac{\partial^2 u}{\partial x \partial t}\right) = 0$$

Their study reveals the following:

(a) the resultant equation is numerically stable if $\frac{h}{\Delta t} > \sqrt{\frac{2}{3}}\, C_{(max)}$, where $C_{(max)}$ is the maximum wave speed $C\sqrt{T'(u_x)}$.

(b) If instead of using the matrix generated by $\left(\frac{\partial u}{\partial t^2}, \frac{\partial u}{\partial t}\right)$ (the so-called consistent mass matrix) one uses a diagonal matrix (the lumped mass approximation) the stability criterion is improved to read $\frac{h}{\Delta t} > \sqrt{2}\, C_{(max)}$.

(c) The finite element solution converges uniformly to the solution, using piecewise linear elements. They do not make any analysis of the order of accuracy of this approach, so this paper does not touch the question raised by Swartz and Wendroff.

4. Questions and Some Answers

A number of questions arise from the intersection of these papers.

1. What is the best method of handling the non-linear terms, and the variable coefficients? Cullen, Hinsman, Oden and Swartz all advocate different answers for different reasons.

2. What does the finite element method do to the gravity waves which worry the people doing finite difference calculations? None of the finite element calculations even mention this question.

3. What are the merits of the "lumped mass" vs "consistent mass" approach which is so familiar to the mechanical engineers?

4. If this were to be implemented as an operational scheme, what are the possible merits of SOR or ADI. Hirsch [17], doing ocean circulation problems, solves the Poisson equation (27) by SOR and the advection equation (29) by ADI. Staniforth and Mitchell [10] use an ADI method for their calculations.

5. Cullen does not integrate by parts to compute his $u \frac{\partial v}{\partial x}$ term. What effect does this have on the solution? Hinsman (unpublished) noted a considerable improvement in his results when he used the weak form (integrated by parts) of the ϕ equation as opposed to the strong form (3).

6. Fix has proved that the vorticity equations for ocean circulation automatically satisfy the desired conservation laws when written as finite elements. Is this also true for the barotropic equations?

7. What does the finite element method do to the phase speed in two dimensions?

We now proceed to answer this last question; at least for a linear model. The finite element approximation to

$$\frac{\partial \phi}{\partial t} + \vec{V}\cdot\nabla\phi = 0 \tag{37}$$

using bilinear elements on rectangles is

$$\frac{kh}{36}\{16\, \dot\phi_{\ell j} + 4[\dot\phi_{\ell,j+1} + \dot\phi_{\ell,j-1} + \dot\phi_{\ell+1,j} + \dot\phi_{\ell-1,j}] + [\dot\phi_{\ell+1,j+1} + \phi_{\ell-1,j+1} +$$

$$\dot\phi_{\ell+1,j-1} + \dot\phi_{\ell-1,j-1}]\} + \frac{3hu}{36}\{[4\, \phi_{\ell+1,j} - 4\, \phi_{\ell-1,j}] + [\phi_{\ell+1,j+1} - \phi_{\ell-1,j+1}]$$

$$+ [\phi_{\ell+1,j-1} - \phi_{\ell-1,j-1}]\} + \frac{3kv}{36}\{[4\, \phi_{\ell,j+1} - 4\, \phi_{\ell,j-1}\} + [\phi_{\ell+1,j+1} -$$

$$\phi_{\ell+1,j-1}] + [\phi_{\ell-1,j+1} - \phi_{\ell-1,j-1}]\} = 0$$

We investigate the phase propagation of this linear model. The exponential solution to (37) is $\phi = A\, e^{-i(\omega t - \hat{k}\cdot\hat{\vec{x}})}$ where $\omega = \vec{V}\cdot\hat{k}$. If we look for ω^* in the approximation

$$\phi_{\ell j}^{*} = A\, e^{-i\omega^{*} t + i\ell k + ijh} \tag{38}$$

then

$$-ikh\omega^{*}\{16 + 8[\cos h + \cos k] + 4 \cos k \cos h\}$$

$$+ 3hui\{\sin k\}(2 + \cos h) + 3kvi\{\sin h\}(2 + \cos k) = 0$$

thus

$$\omega^{*} = \frac{3uh \sin k(2 + \cos h) + 3vk \sin h(2 + \cos k)}{hk(2 + \cos h)(2 + \cos k)}$$

$$= u \frac{3 \sin k}{(2 + \cos k)k} + v \frac{3 \sin h}{(2 + \cos h)h} \tag{39}$$

We then have the following theorem.

Theorem 1: The finite element approximation to (37) using bilinear elements on rectangles has fourth order accuracy in phase speed.

Proof: From (39)

$$\omega^{*} \sim u \frac{\left(3 - \frac{k^2}{2} + 3\,\frac{k^4}{5!} - \ldots\right)}{\left(3 - \frac{k^2}{2} + \frac{k^4}{4!} - \ldots\right)}$$

$$+ \frac{\left(3 - \frac{h^2}{2} + 3\,\frac{h^4}{5!} - \ldots\right)}{\left(3 - \frac{h^2}{2} + \frac{h^4}{4!} - \ldots\right)}$$

$$\therefore \omega^{*} \sim u(1 + 0(k^4)) + v(1 + 0(h^4)) \tag{40}$$

Q.E.D.

This is to be expected, since the basis elements are the tensor product of two one-dimensional linear elements on rectangles.

A more interesting result is whether the same phenomenon happens if we use linear elements on triangles. We have the following result. The finite element approximation for (37) using the general linear element $\phi = a + bx + cy$ on triangles is

$$\frac{kh}{12}\{6\,\dot{\phi}_{\ell,j} + \dot{\phi}_{\ell-1,j} + \dot{\phi}_{\ell,j+1} + \dot{\phi}_{\ell+1,j} + \dot{\phi}_{\ell,j-1} + \dot{\phi}_{\ell+1,j+1} + \dot{\phi}_{\ell-1,j-1}\} +$$

$$\frac{uh}{6}\{2[\phi_{\ell+1,j} - \phi_{\ell-1,j}] + [\phi_{\ell+1,j+1} - \phi_{\ell,j+1}] + [\phi_{\ell,j-1} - \phi_{\ell-1,j-1}]\} + \quad (41)$$

$$\frac{vk}{6}\{2[\phi_{\ell,j+1} - \phi_{\ell,j-1}] + [\phi_{\ell+1,j+1} - \phi_{\ell+1,j}] + [\phi_{\ell-1,j} - \phi_{\ell-1,j-1}]\} = 0 .$$

This also has a fourth order phase speed accuracy as we now show. If we again look at $\phi_{\ell j}{}^*$ as in (38) we get, for the first term in (41)

$$-\frac{2ikh\omega^*}{12}\{3 + \cos k + \cos h + \cos k \cos h - \sin k \sin h\}$$

$$\sim -\frac{2ikh\omega^*}{12}\{6 - k^2 - h^2 - kh + \frac{hk^3}{3!} + \frac{kh^3}{3!} + \ldots\}$$

while the second term is

$$\frac{2ihu}{6}\{3 \sin k - \sin h(1 - \cos k) - \sin k(1 - \cos h)\}$$

$$\sim \frac{2ikhu}{12}\{6 - k^2 - h^2 - kh + \frac{h^3k}{3} + \ldots\}$$

and the third term is

$$\frac{2ikv}{6}\{3 \sin h - \sin k(1 - \cos h) - \sin h(1 - \cos k)\}$$

$$\sim \frac{2ikhv}{12}\{6 - k^2 - h^2 - kh + \frac{k^3h}{2} + \ldots\}$$

Thus

$$\omega^* \sim u(1 + O(h^m k^n)) + v(1 + O(h^m k^n)) \tag{42}$$

where $m + n = 4$. Thus we have shown

Theorem 2: The finite element approximation to (37) using linear elements on triangles has fourth order accuracy in phase speed.

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